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1) Sea $B = B(p,r)$ la bola abierta de centro p y radio $r>0$ en $\mathbb{R}^2$ y $f: B \to \mathbb{R}$ una función diferenciable tal que $\nabla f(x,y) = 0$ $\forall (x,y) \in B$. Probar q f es constante en B.

2) Sea A un abierto de $\mathbb{R}^2$, $p \in A$ y $f: A \to \mathbb{R}$ una función diferenciable en p. Probar q f es continua en p.

3) Sean $g: \mathbb{R}^2 \to \mathbb{R}$ y $h: \mathbb{R} \to \mathbb{R}$ diferenciables. Llamemos $f = h \circ g$ a su composición y sea $p \in \mathbb{R}^2$ un punto crítico de g.

a) Probar q p es un punto crítico de f.

b) Si la matriz Hessiana $H_g(p)$ es definida positiva (o sea $\det(H_g(p)) > 0$ y $g_{xx}(p) > 0$) y $g(p)$ no es un punto crítico de h, decidir la naturaleza de p como punto crítico de f.

4) Sea $u: \mathbb{R}^2 \to \mathbb{R}$ de clase C^1. Probar que

$$\int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(x,y) \cdot \sin(y)\, dy\, dx = \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_y(x,y) \cos(y)\, dy\, dx$$

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① $B_r(P)$ bola abierta $f: B \to \mathbb{R}$ diferenciable $\nabla f(x,y)=0$ $\forall (x,y)$

Probar que f es la función constante.

Sea $(x,y) \in B_r(P)$

$$\alpha: [0,1] \to \mathbb{R}^2$$
$$\alpha(t) = P + t(Q-P)$$

$$f(Q)-f(P) = \langle \nabla f(\underbrace{\alpha(t)}_{\in B_r(P)}), Q-P \rangle$$

$\langle a,b \rangle = \sum_{i=1}^{n} a_i b_i$

$\langle \nabla f \cdot Q-P \rangle$

$f_x \cdot (Q-P)_1 + f_y \cdot (Q-P)_2$

$$\Rightarrow f(Q)-f(P) = \langle 0, Q-P \rangle = 0 \Rightarrow f(Q) = f(P).$$

② $A \subseteq \mathbb{R}^2$ abierto $P \in A$. $f: A \to \mathbb{R}$ una función diferenciable en P. Probar que f es continua

$$0 \leq |f(x)-f(P)| \leq |f(x)-f(P) - \nabla f(x)(x-P) + \nabla f(P)(x-P)|$$
$$\leq |f(x)-f(P) - \nabla f(x-P)| + \|\nabla f(P)\| \cdot |(x-P)|$$
$$= \frac{|f(x)-f(P) - \nabla f(x-P)|}{\|x-P\|} \|x-P\| + \|\nabla f(P)\| \, |x-P|$$

$$\lim_{x\to P} f(x)-f(P) \leq \lim_{x \to P} \underbrace{\frac{f(x)-f(P)-\nabla f(x-P)}{\|x-P\|}}_{\text{Acotado}} \|x-P\| + \|\nabla f(P)\| \, \|x-P\| \leq 0$$

infinit

$$\boxed{\Rightarrow \lim_{x\to P} f(x) = f(P)}$$

③ $\left.\begin{array}{l} g: \mathbb{R}^2 \to \mathbb{R} \\ h: \mathbb{R} \to \mathbb{R} \end{array}\right\}$ diferenciables $f = h \circ g$ $P \in \mathbb{R}^2$ pto crítico de g.

a) Probar que P es un punto crítico de f.

b) H_g def positiva $(\det(H_g(P)) > 0$ y $g_{xx} > 0)$

$g(P)$ no es pto crítico de h. Decidir la naturaleza de P como pto crítico de f.

a) $\nabla f = \nabla_{h \circ g} = Dh(g(x)) \, \nabla g(x)$
$= h'(g(x)) \cdot \nabla g(x)$ $\Rightarrow$ $Df(P) = h'(g(P)) \cdot \nabla g(P)$
$= h'(g(P)) \cdot 0$

(b) $\underset{1\times 2}{\nabla f} = \underset{1\times 1}{h'(g(x))} \cdot \underset{1\times 2}{\nabla g(x)}$

$\begin{cases} f_x = h'(g(x,y)) \cdot g_x \\ f_y = h'(g(x,y)) \cdot g_y \end{cases}$

(*) $\Rightarrow \det(H_f) > 0$

$\frac{\partial^2 f}{\partial xx} =$

$Hf(x) = D_{h'(g(x)) \cdot \nabla g(x)} = h''(g(x)) \cdot (\nabla g(x)) \nabla g(x) + h'(g(x)) \cdot H_g(x)$

$f_{xx} = h''(g(x,y)) \cdot (g_x)^2 + h'(g(x,y)) \cdot g_{xx}$

$f_{yy} = h''(g(x,y)) \cdot (g_y)^2 + h'(g(x,y)) \cdot g_{yy}$

$f_{xy} = h''(g(x,y)) \cdot g_y \cdot g_x + h'(g(x,y)) \cdot g_{xy}$

$f_{yx} = h''(g(x,y)) \cdot g_y \cdot g_x + h'(g(x,y)) \cdot g_{yx}$

$f_{xx} = \underbrace{h''(g(x,y)) \cdot (g_x)^2}_{0} + \underbrace{h'(g(x,y))}_{\neq 0} \underbrace{g_{xx}}_{\geq 0}$

?

$\Rightarrow$ P es extremo (no es pto silla) de f.

$\det(H_f) = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx}$

$= (h''(g(x,y)) \cdot (g_x)^2 + h'(g(x,y)) \cdot g_{xx}) \cdot (h''(g(x,y)) \cdot (g_y)^2 + h'(g(x,y)) \cdot g_{yy})$

$= h''(g)(g_x)^2 h''(g)(g_y)^2 + h''(g)\, h'(g) \cdot g_{yy} \cdot g(x^2) + (h''(g))^2 \cdot (g_y)^2 \cdot g_{xx} +$ $(h'(g))$

$(h'(g))^2 g_{xx}\, g_{yy}$

$-$

$(h''(g) \cdot g_y g_x + h'(g) \cdot g_{xy})(h''(g) \cdot g_y \cdot g_x + h'(g) \cdot g_{yx})$

$= (h''(g))^2 (g_y)^2 (g_x)^2 + h''(g) \cdot h'(g) \cdot g_{yx} \cdot g_y \cdot g_x + h''(g) h'(g) \cdot g_{xy} \cdot g_x \cdot g_y$

$+ (h'(g))^2 \cdot g_{xy} \cdot g_{yx}$

$\cancel{(h'')^2 (g_x)^2 (g_y)^2} + (h'') \cdot (h') (g_{yy})(g_x)^2 + (h'')(h')(g_y)^2 (g_{xx}) + (h')^2 \cdot (g_{xx})(g_{yy})$

$\cancel{+ (h'')^2 (g_y)^2 (g_x)^2} - (h'')(h')(g_{yx})(g_x)(g_y) - (h'') \cdot (h')(g_{xy})(g_x)(g_y) - (h')^2 (g_{xy})(g_{yx})$

$(h'')(h')\big((g_x)^2 (g_{yy}) + (g_y)^2 (g_{xx}) - (g_{yx})(g_x)(g_y) - (g_{xy})(g_x)(g_y)\big) + (h')^2 \underbrace{\big((g_{xx})(g_{yy}) - (g_{xy})(g_{yx})\big)}_{>0}$ >0

$(h'')(h')\big(\underbrace{(g_x)^2}_{0}(g_{yy}) + \underbrace{(g_y)^2}_{0}(g_{xx}) - \underbrace{((g_x)(g_y))}_{0}(g_{xy} + g_{yx})\big) = 0 + \geq 0$ (*)

(como P es pto crítico de g)

(4) $u:\mathbb{R}^2 \to \mathbb{R} \in \mathcal{C}^1 \Rightarrow \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(x,y).\, \sin(y)\, dy\, dx =$

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$$\int_0^1 \int_{-\pi/2}^{\pi/2} u_y(x,y).\, \cos(y)\, dy\, dx$$

$y = \cos^{-1}(v)$
$dy =$

$U = U(x,y)$
$v = \cos(y)$

"du" $u_y = U_y$
"dv" $v_y = -\operatorname{sen}(y)$

$$\int_0^1 \left(\int_{-\frac{\pi}{2}}^{\pi/2} u(x,y).\, \operatorname{sen}(y).\, dy \right) dx = \int_0^1 \int_{-\frac{\pi}{2}}^{\pi/2} u_y(x,y).\, \cos(y)\, dy\, dx$$

$V = \cos y$
$dV = -\sin y \cdot dy$

$$\int \underbrace{u(x,y)}_{u}.\, \underbrace{\operatorname{sen}(y)}_{v'}\, dy = u.v - \int u' v\, dy$$

$$= \left(u(-1).\cos(y) - \int u_y.(x,y)(-1)\cos(y) \right)' = u(x,y).\, \operatorname{sen}(y)$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\pi/2} u(x,y).\, \operatorname{sen}(y)\, dy = \left(-u(x,y).\cos(y) + \int u_y(x,y).\cos(y)\, dy \right) \Big|_{y=-\pi/2}^{y=\pi/2}$$

$$\int_0^1 \left(\left(-u(x,y).\cos(y) + \int U_y(x,y).\cos(y)\, dy \right) \Big|_{y=-\pi/2}^{y=\pi/2} \right) dx$$

$\int Abc\ dy$

dx

$$\int_0^1 \left(-u(x,y).\cos(y) + \int U_y(x,y).\cos(y)\, dy \Big|_{y=\frac{\pi}{2}} - \left(-u(x,y).\cos(y) + \int U_y(x,y).\cos(y)\, dx \right|_{y=-\frac{\pi}{2}} \right)$$

$$\int_0^1 \cancel{-U\left(x,\tfrac{\pi}{2}\right).\cos\left(\tfrac{\pi}{2}\right)} + \left(\int U_y(x,y).\cos(y)\, dy \right) \Big|_{y=\frac{\pi}{2}} + \cancel{u\left(x,-\tfrac{\pi}{2}\right).\cos\left(-\tfrac{\pi}{2}\right)} - \left(\int U_y(x,y).\cos(y)\, dy \right) \Big|_{y=-\frac{\pi}{2}}$$

$$\int_0^1 \left(-\left(\int U_y(x,y).\cos(y)\, dy \right) \Big|_{y=-\frac{\pi}{2}} + \left(\int U_x(x,y).\cos(y)\, dy \right) \Big|_{y=\frac{\pi}{2}} \right) dx$$

$$= \int_0^1 \int_{-\pi/2}^{\pi/2} U_y(x,y).\cos(y)\, dy\, dx = \int_0^1 \int_{-\frac{\pi}{2}}^{\pi/2} U_y(x,y).\cos(y)\, dy\, dx$$

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