1) Sea $f: R^2 \to R^2$ definida como $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Probar que *f* es diferenciable en el $(0,0)$ pero las derivadas parciales no son continuas en ese punto.

$$\frac{\partial f}{\partial x}(0,0) = \underset{h\to 0.}{lím}\frac{f(h,0)}{h} = (h^2)\sin\left(\frac{1}{\sqrt{h^2}}\right)$$

$$= (h^2)\sin\left(\frac{1}{|h|}\right) = \frac{h^2}{|h|} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \underset{h\to 0.}{lím}\frac{f(0,h)}{h} = (h^2)\sin\left(\frac{1}{\sqrt{h^2}}\right) = 0$$

$$\underset{(x,y)\to(0,0)}{lím}\frac{\|(xy)\|^2\sin\left(\frac{1}{\|(xy)\|}\right)}{\|(xy)\|} \leq \underset{(x,y)\to(0,0)}{lím}\frac{\|(xy)\|^2 1}{\|(xy)\|} = \|(xy)\| \to 0$$ (es diferenciable)

$$\frac{\partial f}{\partial x} = (2x)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\frac{-2x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial x}(x,y) = (2x)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\frac{-2x}{\sqrt{x^2+y^2}}$$

$$\underset{(x,y)\to(0,0)}{lím}(2x)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\frac{-2x}{\sqrt{x^2+y^2}}$$

$$\underset{(x,x)\to(0,0)}{lím}\cos\left(\frac{1}{\sqrt{2}|x|}\right)\frac{-2x}{\sqrt{2}|x|}$$

Oscila

2) Probar que $\sin(x)\sin(y)\sin(z) \leq \frac{1}{8}$ si $0 \leq x, y \leq \pi_2$, $x+y+z = \frac{\pi}{2}$.

$$\sin(x)\sin(y)\sin(z) \leq \frac{1}{8}$$

$$x+y+z = \frac{\pi}{2}$$

$$0 \leq x, y \leq \pi$$

$$-\frac{3\pi}{2} \leq z \leq \frac{\pi}{2}$$

Calculo máximos con Lagrange

$$f(x,y) = \sin(x)\sin(y)\sin(z)$$

$$g(x,y) = x + y + z - \frac{\pi}{2}$$

$$\frac{\partial f}{\partial x} = \cos(x)\sin(y)\sin(z)$$

$$\frac{\partial f}{\partial y} = \sin(x)\cos(y)\sin(z)$$

$$\frac{\partial f}{\partial z} = \sin(x)\sin(y)\cos(z)$$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 1$$

$$\begin{cases} \cos(x)\sin(y)\sin(z) = \lambda \\ \sin(x)\cos(y)\sin(z) = \lambda \\ \sin(x)\sin(y)\cos(z) = \lambda \\ x + y + z = \frac{\pi}{2} \end{cases}$$

$$\cos(x)\sin(y)\sin(z) = \sin(x)\cos(y)\sin(z)$$

$$\tan(y) = \tan(x)$$

$$x = y + k\pi$$

O bien

$$\sin(z) = 0$$

$$z = 0 \vee z = -\pi$$

$$\sin(x)\cos(y)\sin(z) = \sin(x)\sin(y)\cos(z)$$

$$\tan(z) = \tan(y)$$

$$z = y + k\pi$$

O bien

$$\sin(x) = 0$$

$$x = 0 \vee x = \pi$$

Combinaciones posibles:

$$z = 0 \quad x = 0 \quad y = \frac{\pi}{2}$$

$$z = -\pi \quad x = \pi \quad y = \frac{\pi}{2}$$

$z=0 \quad y=0 \quad x=\frac{\pi}{2}$

$z=-\pi \quad y=\pi \quad x=\frac{\pi}{2}$

$x=0 \quad y=0 \quad z=\frac{\pi}{2}$

$x=0 \quad y=\pi \quad z=-\frac{\pi}{2}$

$x=\pi \quad y=\pi \quad z=-\frac{3\pi}{2}$

$x=\pi \quad y=0 \quad z=-\frac{\pi}{2}$

Hasta acá, todos dan 0 porque incluyen un seno de 0 o de π .

$z=y+q\pi$

$x=y+r\pi$

$y+q\pi+y+y+r\pi=\frac{\pi}{2}$

$k=q+r$

$3y+k\pi=\frac{\pi}{2}$

$y=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}$

$x=y$

$z=\frac{\pi}{2}-2y=\frac{\pi}{6},-\frac{\pi}{2},-\frac{7\pi}{6}$

$\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)=\frac{1}{8}$(máximo)

$\sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{5\pi}{6}\right)\sin\left(-\frac{7\pi}{6}\right)=\frac{1}{8}$ (máximo)

$\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right)=-1$ (mínimo)

3) Sea $f:[a,b]\to[\alpha,\beta]$ biyectiva de clase C^2 con inversa también de clase C^2 $g:[\alpha,\beta]\to[a,b]$ (se verifica $f\circ g(x)=x$ y $g\circ f(x)=x$).

a) Calcular $g''(x)$ para todo $x\in(\alpha,\beta)$ en función de *f* y sus derivadas.

$$g(f(x)) = x$$
$$g'(f(x))f'(x) = 1$$
$$g'(f(x)) = \frac{1}{f'(x)}$$
$$g''(f(x))f'(x) = -\frac{f''(x)}{(f'(x))^2}$$
$$g''(f(x)) = -\frac{f''(x)}{(f'(x))^3}$$

$$g''(x) = -\frac{f''(g(x))}{(f'(g(x)))^3}$$

b) Si $f'(x) > 0$, $f''(x) > 0$ para todo $x \in (a,b)$, probar que $g''(x) < 0$ para todo $x \in (\alpha, \beta)$.

$$g''(x) = -\frac{f''(g(x))}{(f'(g(x)))^3}$$

$$g(x) \in (a,b)$$
$$f''(g(x)) > 0$$
$$f'(g(x)) > 0$$
$$(f'(g(x)))^3 > 0$$
$$\frac{f''(g(x))}{(f'(g(x)))^3} > 0$$
$$-\frac{f''(g(x))}{(f'(g(x)))^3} < 0$$
$$g''(x) < 0$$

4) Sea A un abierto de R^2, $f : A \to R$ diferenciable, $p = (x_0, y_0) \in A$, $v_1 = \frac{1}{\sqrt{5}}(1,2)$, $v_2 = \frac{1}{\sqrt{13}}(2,3)$ tal que $\frac{\partial f}{\partial v_1}(p) = a$ y $\frac{\partial f}{\partial v_2}(p) = b$.

a) Calcular $\nabla f(p)$.

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p) \right)$$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$F_2 - \frac{2\sqrt{5}}{\sqrt{13}} F_1 \rightarrow F_2$$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & \frac{-1}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \end{pmatrix} = \begin{pmatrix} a \\ b - \frac{2\sqrt{5}}{\sqrt{13}} a \end{pmatrix}$$

$$F_1 + \frac{2\sqrt{13}}{\sqrt{5}} F_2 \rightarrow F_2$$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \end{pmatrix} = \begin{pmatrix} -3a + \frac{2\sqrt{13}}{\sqrt{5}} b \\ b - \frac{2\sqrt{5}}{\sqrt{13}} a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \end{pmatrix} = \begin{pmatrix} -3\sqrt{5}a + 2\sqrt{13}b \\ 2\sqrt{5}a - \sqrt{13}b \end{pmatrix}$$

$$\frac{\partial f}{\partial x}(p) = -3\sqrt{5}a + 2\sqrt{13}b$$

$$\frac{\partial f}{\partial y}(p) = 2\sqrt{5}a - \sqrt{13}b$$

b) ¿Cuánto valen a y b si f tiene un máximo local en p?

$$\nabla f(p) = (0,0)$$

$$0 = -3\sqrt{5}a + 2\sqrt{13}b$$

$$0 = 2\sqrt{5}a - \sqrt{13}b$$

$$0 = \sqrt{5}a$$

$$b = 0$$

c) Probar que si $(a,b) \neq (0,0)$ el plano tangente al gráfico de f en p nunca es horizontal.

Plano tangente:

$$z = f(p) + \frac{\partial f}{\partial x}(p)(x - x_0) + \frac{\partial f}{\partial y}(p)(y - y_0)$$

$$z = f(p) - \frac{\partial f}{\partial x}(p)x_0 - \frac{\partial f}{\partial y}(p)y_0 + \frac{\partial f}{\partial x}(p)x + \frac{\partial f}{\partial y}(p)y$$

$$f(p) - \frac{\partial f}{\partial x}(p)x_0 - \frac{\partial f}{\partial y}(p)y_0 = cte$$

$$z = cte + \frac{\partial f}{\partial x}(p)x + \frac{\partial f}{\partial y}(p)y$$

Pero para que el plano tangente sea horizontal, $z = cte$. Es decir que $\frac{\partial f}{\partial x}(p)x = 0$ y $\frac{\partial f}{\partial y}(p)y = 0$. Pero no pueden ser ambas derivadas parciales 0 porque si $(a,b) \neq (0,0)$, $\nabla f(p) \neq (0,0)$. Para que sea $(0,0)$, tendría que cumplirse la condición del punto anterior, es decir, $\begin{matrix} 0 = \sqrt{5}a \\ b = 0 \end{matrix}$. Luego, $\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right) \neq (0,0)$ y el plano tangente no es horizontal.

5) Considerar la función $F : R^2_{\geq 0} \rightarrow R$ dada por $F(x,y) = \iint_{R_{xy}} e^{(u-1)(v^2-v)} du dv$, donde R_{xy} es el rectángulo dado por $[0,x] \times [0,y]$.

a) Calcular $f(x,0)$ y $f(0,y)$ para todos los valores de $x, y \geq 0$.

Es 0, estás integrando sobre una superficie de área 0.

b) Encontrar $\nabla f(1,1)$

$$\left(\int_a^x f(v)dv\right)' = f(x)$$

$$\frac{\partial f}{\partial x}\left(\int_0^x \left(\int_0^y e^{(u-1)(v^2-v)} du\right) dv\right) =$$

$$\int_0^y e^{(u-1)(v^2-v)} du = \int_0^y \left(e^{(v^2-v)}\right)^{(u-1)} du$$

$$\int \left(e^{(v^2-v)}\right)^{(u-1)} du$$

$$\int b^x dx = \frac{b^x}{\ln b}$$

$$= \frac{\left(e^{(v^2-v)}\right)^{(u-1)}}{v^2 - v}$$

$$= \frac{e^{(v^2-v)(y-1)} - e^{(-v^2+v)}}{v^2 - v}$$

$$\frac{\partial f}{\partial x}\left(\int_0^x\left(\int_0^y e^{(u-1)\left(v^2-v\right)}du\right)dv\right)=$$

$$\frac{\partial f}{\partial x}\left(\int_0^x \frac{e^{\left(v^2-v\right)(y-1)}-e^{\left(-v^2+v\right)}}{v^2-v}dv\right)=\frac{e^{\left(x^2-x\right)(y-1)}-e^{\left(-x^2+x\right)}}{x^2-x}$$

$$\frac{\partial f}{\partial y}\left(\int_0^y\left(\int_0^x e^{(u-1)\left(v^2-v\right)}du\right)dv\right)=$$

$$\int_0^x e^{(u-1)\left(v^2-v\right)}du=$$

$$t=\left(v^2-v\right)(u-1)$$

$$dt=\left(v^2-v\right)du$$

$$\int_0^x \frac{e^{\left(v^2-v\right)(u-1)}}{v^2-v}=\frac{e^{\left(v^2-v\right)(x-1)}}{v^2-v}-\frac{e^{\left(-v^2+v\right)}}{v^2-v}$$

$$\frac{\partial f}{\partial y}\left(\int_0^y\left(\frac{e^{\left(v^2-v\right)(x-1)}}{v^2-v}-\frac{e^{\left(-v^2+v\right)}}{v^2-v}\right)dv\right)=\frac{e^{\left(y^2-y\right)(x-1)}-e^{\left(-y^2+y\right)}}{y^2-y}$$

$$\nabla f(x,y)=\frac{e^{\left(x^2-x\right)(y-1)}-e^{\left(-x^2+x\right)}}{x^2-x},\frac{e^{\left(y^2-y\right)(x-1)}-e^{\left(-y^2+y\right)}}{y^2-y}$$

$$\underset{(x,y)\to(1,1)}{lím}\frac{e^{\left(x^2-x\right)(y-1)}-1}{\left(x^2-x\right)(y-1)}(y-1)+\underset{(x,y)\to(1,1)}{lím}\frac{1-e^{\left(-x^2+x\right)}}{x^2-x}=\underset{(x,y)\to(1,1)}{lím}\frac{1-e^{\left(-x^2+x\right)}}{x^2-x}$$

$$\underset{(x,y)\to(1,1)}{lím}\frac{e^{-\left(x^2-x\right)}-1}{-\left(x^2-x\right)}=1$$

$$\nabla f(1,1)=(1,1)$$