ORI

1/7

1	2	3	4	5
B	B-	B-	B	B

CALIF.
A.

APELLIDO Y NOMBRE: [redacted]

LIBRETA: [redacted]

TEMA 1

TURNO PRÁCTICA: ☒ Mañana ☐ Tarde ☐ Noche

Álgebra I - 2do. Cuatrimestre de 2014
Segundo parcial
02/12/2014

1. Calcular el resto de la división de $3^{3+11^{527}}$ por 143.

2. Sea $n \in \mathbb{N}$ tal que
$$\frac{(7-7i)^{5n+2}}{-1+i}$$
es un número real negativo. Calcular todos los posibles restos de dividir n por 8.

3. Para cada $w \in G_5$ calcular
$$w^{13} + 3w^{72} + \overline{w}^{34} + w^{-71} - 2(\overline{w}^{22})^{-1} + 2$$

4. Factorizar en $\mathbb{C}[X]$ el polinomio $f = X^4 + (4-4i)X^3 - 18iX^2 - (16+24i)X - (16+8i)$ sabiendo que f tiene una raíz $\alpha \in \mathbb{R}$ doble.

5. Sean α, β y γ las raíces de $X^3 - 3X^2 + 4X - 5$. Hallar un polinomio mónico de grado 3 cuyas raíces sean $1 - \frac{1}{\alpha}$, $1 - \frac{1}{\beta}$ y $1 - \frac{1}{\gamma}$.

Complete esta hoja con sus datos y entréguela con el resto del examen.
Justifique todas sus respuestas.

HOJA Nº 2/7

FECHA

1) $r_{143}(3^{3+11^{527}}) = x \Rightarrow 3^{3+11^{527}} \equiv x \ (143)$

$143 = 11 . 13$

$$3^{3+11^{527}} \equiv x \ (143) \xRightarrow{T.CH.R} \begin{cases} 3^{3+11^{527}} \equiv x_1 (11) \\ 3^{3+11^{527}} \equiv x_2 (13) \end{cases}$$

RESUELVO CADA ECUACIÓN POR SEPARADO

$\underline{3^{3+11^{527}} \equiv x_1 (11)}$

$(3:11) = 1$ y 11 primo $\xRightarrow{P.T.F} 3^{10} \equiv 1 \ (11)$

$3 + 11^{527} = 10q + r_{10}(3+11^{527})$

$$3^{3+11^{527}} = (3^{10})^q . 3^{r_{10}(3+11^{527})} \underset{(11)}{\equiv} 3^{r_{10}(3+11^{527})}$$

$3+11^{527} = 10q + r_{10}(3+11^{527}) \Leftrightarrow 3+11^{527} \equiv r_{10}(3+11^{527}) \ (10)$

$$3+11^{527} \underset{(10)}{\equiv} 3 + (1)^{527} = 3+1 = 4$$

$$3+11^{527} \equiv 4 \ (10)$$

$$3^{r_{10}(3+11^{527})} = 3^4 = 81 \underset{(11)}{\equiv} 4$$

$$\boxed{3^{3+11^{527}} \equiv 4 \ (11)}$$

NOTA

$$3^{3+11^{527}} \equiv x_2\ (13)$$

$(3;13)=1$ y 13 es primo $\overset{\text{P.T.F}}{\Longrightarrow}$ $3^{12} \equiv 1\ (13)$ ✓

$$3+11^{527} = 12\tilde{q} + \tilde{r}_{12}(3+11^{527})$$ ✓

$$3^{3+11^{527}} = (3^{12})^{\tilde{q}} \cdot 3^{\tilde{r}_{12}(3+11^{527})} \underset{(13)}{\equiv} 3^{\tilde{r}_{12}(3+11^{527})}$$ ✓

$$3+11^{527} = 12\tilde{q} + \tilde{r}_{12}(3+11^{527}) \Leftrightarrow 3+11^{527} \equiv \tilde{r}_{12}(3+11^{527})\ (12)$$

$$3+11^{527} \underset{(12)}{\equiv} 3+(-1)^{527} = 3-1 = 2$$ ✓

$$3+11^{527} \equiv 2\ (12)$$ ✓

$$3^{\tilde{r}_{12}(3+11^{527})} = 3^2 \equiv 9\ (13)$$ ✓

$$\left\{\begin{array}{l} 3^{3+11^{527}} \equiv 4\ (11) \\ 3^{3+11^{527}} \equiv 9\ (13) \end{array}\right. \overset{\text{T.CH.R}}{\Longrightarrow} \boxed{3^{3+11^{527}} \equiv 48\ (143)}$$ ✓

ya que 11 y 13 coprimos

RTA: $r_{143}(3^{3+11^{527}}) = 48$ ✓

NOTA

HOJA N° 3/7

FECHA

2) $n \in \mathbb{N}$

$$\frac{(7-7i)^{5n+2}}{-1+i}$$ ES UN NUMERO REAL NEGATIVO

CALCULAR TODOS LOS POSIBLES RESTOS DE $r_8(n)$

ME FIJO LOS ARGUMENTOS $a \in (-\alpha, 0)$ ✓ $\downarrow -\infty$

$$\arg\left((7-7i)^{5n+2}\right) - \arg(-1+i) = \arg(a)$$

$$(5n+2)\left(\frac{7}{4}\pi\right) - 2k\pi - \frac{3}{4}\pi = \pi \quad ✓$$

ATENCION CON LOS PARENTESIS

$$\left(\frac{7}{4}\right)(5n+2) = 1 + \frac{3}{4} + 2k\pi$$

$$\left(\frac{7}{4}\right)(5n+2) = \frac{7+8k}{4}$$

$$35n + 14 = 7 + 8k \iff$$

$$\iff \quad 35n + 14 \equiv 7 \ (8) ✓$$

$$\iff \quad 5n + 2 \equiv 1 \ (8)$$

$(7:8) = 1$

DIVIDO POR 7 → En realidad multiplico por el inverso de 7

$$\iff \quad 5n \equiv 7 \ (8)$$, COMO $(5:8) = 1$ Y $1 | 7$ TIENE SOL LA ECUACIÓN Y ES UNICA

$$\boxed{n \equiv 3 \ (8)}$$

¿cómo que es única?
Hay infinitos $n \in \mathbb{N}$ que cumplen

RTA $\boxed{r_8(n) = 3}$ ✓

NOTA

HOJA Nº 4/7

FECHA

3) $\omega \in G_5$

$$\omega^{13} + 3\omega^{72} + \overline{\omega}^{34} + \omega^{-71} - 2(\overline{\omega}^{22})^{-1} + 2$$

$\omega^{13} = (\omega^5)^2 . \omega^3 = 1 . \omega^3 = \omega^3$ ✓

$\omega \in G_5 \Rightarrow \omega^5 = 1$ ✓

$\omega^{72} = (\omega^5)^{14} . \omega^2 = 1 . \omega^2 = \omega^2$ ✓

$\omega \in G_5 \Rightarrow \omega^5 = 1$

¿Porqué vale esta igualdad?

$\overline{\omega}^{34} = \overline{\omega^{34}} = \overline{\omega^4} = \overline{\omega}^4 = \omega^{5-4} = \omega$ ✓

$\omega^{34} = (\omega^5)^6 . \omega^4 = \omega^4$

$\omega^{-71} = \omega^{-1} = \omega^{5-1} = \omega^4$ ✓

$(\overline{\omega}^{22})^{-1} = \overline{\omega}^{-22} = \overline{\omega}^{-2} = \omega^{5-(-2)} = \omega^7 = \omega^2$ ✓

REEMPLAZANDO TODO

$$\omega^3 + 3\omega^2 + \omega + \omega^4 - 2\omega^2 + 2 \ ✓$$

$\Leftrightarrow \quad \omega^3 + \omega^2 + \omega + \omega^4 + 2$

$\Leftrightarrow \quad \underbrace{1 + \omega + \omega^2 + \omega^3 + \omega^4}_{} + 1$

$$\sum_{i=0}^{4} \omega^i = \frac{\omega^5 - 1}{\omega - 1} = \frac{1-1}{\omega - 1} = 0 \ ✓$$

$\omega \neq 1$ $\qquad$ $\omega \in G_5 \Rightarrow \omega^5 = 1$

$\underline{\omega = 1}$

$$1 + (1) + (1)^2 + (1)^3 + (1)^4 + 1 = 6 \ ✓$$

$\underline{\omega \neq 1}$

$$\underbrace{1 + \omega + \omega^2 + \omega^3 + \omega^4}_{} + 1 = 0 + 1 = 1 \ ✓$$

NOTA

$$\omega^{13} + 3\omega^{72} + \bar{\omega}^{34} + \omega^{-71} - 2(\bar{\omega}^{22})^{-1} + 2 = \begin{cases} 6 & \text{si } \omega = 1 \\ 1 & \text{si } \omega \neq 1 \end{cases}$$

NOTA

HOJA N° 5/7

FECHA

4) $f = x^4 + (4-4i)x^3 - 18ix^2 - (16+24i)x - (16+8i)$

$\alpha \in \mathbb{R}$, raiz doble

como $\alpha \in \mathbb{R}$ es raiz doble $\Rightarrow f(\alpha)=0$ y $f'(\alpha)=0$ ✓

$$\begin{aligned} f(\alpha) &= \alpha^4 + (4-4i)\alpha^3 - 18i\alpha^2 - (16+24i)\alpha - (16+8i) \\ &= \alpha^4 + 4\alpha^3 - 4i\alpha^3 - 18i\alpha^2 - 16\alpha - 24\alpha i - 16 - 8i \\ &= (\alpha^4 + 4\alpha^3 - 16\alpha - 16) + (-4\alpha^3 - 18\alpha^2 - 24\alpha - 8)i \\ &= \underbrace{(\alpha^4 + 4\alpha^3 - 16\alpha - 16)}_{=0} - 2\underbrace{(2\alpha^3 + 9\alpha^2 + 12\alpha + 4)}_{=0}i \end{aligned}$$

Es más fácil buscar las raíces reales de f' y ver cuáles son, además, raíces de f.

Saco las raices de $\underbrace{2\alpha^3 + 9\alpha^2 + 12\alpha + 4}_{g(\alpha)}$

$g(\alpha) \in \mathbb{Q}[x] \Rightarrow$ busco raices con el teorema Gauss ✓

$P = \{\pm 1, \pm 2, \pm 4\}$ $\qquad q = \{\pm 1, 2\}$

$\frac{p}{q} = \{\pm 1, \pm \frac{1}{2}, \pm 2, \pm 4\}$

$\beta \in \frac{p}{q}, \beta > 0 \Rightarrow g(\beta) = \underbrace{2\beta^3}_{>0} + 9\underbrace{\beta^2}_{>0} + 12\underbrace{\beta}_{>0} + \underbrace{4}_{>0} \neq 0$ ✓

Por lo tanto $\frac{p}{q} = \{-1, -\frac{1}{2}, -2, -4\}$ ✓

$g(-1) = -2 + 9 - 12 + 4 = -1 \neq 0$

$g(-2) = 2(-8) + 9(4) + 12(-2) + 4 = 0 \Rightarrow -2$ es raiz

$g(-4) = 2(-64) + 9(16) + 12(-4) + 4 = -28 \neq 0$

NOTA

$g\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{8}\right) + 9\left(\frac{1}{4}\right) + 12\left(-\frac{1}{2}\right) + 4 = 0$ $\quad -\frac{1}{2}$ TAMBIEN ES RAIZ

$\underbrace{(\alpha+2)(\alpha+1/2)} \mid 2\alpha^3 + 9\alpha^2 + 12\alpha + 4$

multiplico por 2 $\quad \alpha^2 + \frac{5}{2}\alpha + 1$

$2\alpha^2 + 5\alpha + 2$

$$\begin{array}{l|l} 2\alpha^3 + 9\alpha^2 + 12\alpha + 4 & 2\alpha^2 + 5\alpha + 2 \\ -\ 2\alpha^3 + 5\alpha^2 + 2\alpha & \alpha + 2 \\ \hline \quad 4\alpha^2 + 10\alpha + 4 & \\ -\ 4\alpha^2 + 10\alpha + 4 & \\ \hline \quad \boxed{0} & \end{array}$$

No! +

$g(\alpha) = (\alpha+2)^2(\alpha - 1/2) \cdot 2$

$2\alpha^3 + 9\alpha^2 + 12\alpha + 4$ SEANULA EN $\alpha = -2$ y $\alpha - \frac{1}{2}$ ✓

VEO SI VERIFICA CON $\alpha^4 + 4\alpha^3 - 16\alpha - 16$

$\underline{\alpha = -2}$

$(-2)^4 + 4(-2)^3 - 16(-2) - 16 = 0$ ✓

$\underline{\alpha = -1/2}$

$\left(-\frac{1}{2}\right)^4 + 4\left(-\frac{1}{2}\right)^3 - 16\left(-\frac{1}{2}\right) - 16 = \frac{1}{16} - \frac{1}{2} + 8 - 16 = \frac{-135}{16} \neq 0$

$f(\alpha) = 0 \underset{\alpha \in \mathbb{R}}{\Leftrightarrow} \alpha = -2$ ✓

$f'(x) = 4x^3 + 3(4-4i)x^2 - 36ix - (16+24i)$ ✓

$f'(-2) = 4(-2)^3 + 3(4-4i)(4) - 36i(-2) - 16 - 24i$

$= \cancel{-32} + \cancel{48} - \cancel{48i} + \cancel{72i} - \cancel{16} - \cancel{24i} = 0$ ✓

NOTA

HOJA N° 6/7

FECHA

$\alpha = -2 \in \mathbb{R}$ y es raiz doble ✓

$\underbrace{(x+2)^2}_{x^2+4x+4} \mid x^4 + (4-4i)x^3 - 18ix^2 - (16+24i)x - (16+8i)$ ✓

$$
\begin{array}{l|l}
x^4 + (4-4i)x^3 - 18ix^2 - (16+24i)x - (16+8i) & x^2+4x+4 \\
\cline{2-2}
-\ \underline{x^4 + 4x^3 + 4x^2} & x^2 - 4ix + (-4-2i) \\
-4ix^3 + (-4-18i)x^2 - (16+24i)x - (16+8i) & \\
-\ \underline{-4ix^3 - 16ix^2 - 16ix} & \\
(-4-2i)x^2 + (-16-8i)x + (-16-8i) & \\
-\ \underline{(-4-2i)x^2 + (-16-8i)x + (-16-8i)} & \\
\boxed{0} &
\end{array}
$$

$$f(x) = (x+2)^2 \underbrace{(x^2 - 4ix + (-4-2i))}_{\text{busco raices}}$$

$x_i = \dfrac{4i + w}{2}$, $\quad w^2 = (-4i)^2 - 4(1)(-4-2i)$ ✓

$w^2 = -16 + 16 + 8i$

$w^2 = 8i$ ✓

$|w|^2 = |8i| = 8 =$ → Es más fácil resolverlo usando binómica

$\boxed{|w| = \sqrt{8}}$ ✓

$\arg(w^2) = \arg(8i) \qquad \arg(w) = \gamma$

$2\gamma - 2k\pi = \dfrac{\pi}{2}$

$\gamma = \dfrac{(4k+1)\pi}{4}$ ✓

$w_k = \sqrt{8}\left(\cos\left(\dfrac{(4k+1)\pi}{4}\right) + i\,\mathrm{sen}\left(\dfrac{(4k+1)\pi}{4}\right)\right) \quad k = 0,1$ ✓

NOTA

$$w_0 = \sqrt{8}\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{SEN}\left(\frac{\pi}{4}\right)\right)$$

$$w_0 = \sqrt{8}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$\boxed{w_0 = 2 + 2i} \checkmark$$

$$w_1 = \sqrt{8}\left(\cos\left(\frac{5}{4}\pi\right) + i\,\text{SEN}\left(\frac{5}{4}\pi\right)\right)$$

$$\boxed{w_1 = -2 - 2i} \checkmark$$

$$x_1 = \frac{4i + 2 + 2i}{2} = \frac{2 + 6i}{2} = 1 + 3i \checkmark$$

$$x_2 = \frac{4i - 2 - 2i}{2} = \frac{-2 + 2i}{2} = -1 + i \checkmark$$

FACTORIZACION $\mathbb{C}[x]$

$$f = (x+2)^2 (x - 1 - 3i)(x + 1 - i) \checkmark$$

COMO SON DE $gr = 1$ SON IRREDUCIBLEZ EN $\mathbb{C}$

NOTA