1,30

ZZ
(5)

PROBABILIDADES Y ESTADÍSTICA (C)
EXAMEN FINAL
(19/03/04)

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EL EXAMEN FINAL SE APRUEBA CON 50 PUNTOS

ENUNCIAR LAS PROPIEDADES QUE SE UTILIZAN

JUSTIFICAR TODAS LAS RESPUESTAS

1. (25 pts.) Sea $T \sim \Gamma(2,2)$ y $U|_{T=t} \sim \mathcal{U}([0, \frac{2}{t}])$.

   (a) Hallar $E(U)$.

   (b) Hallar la función de densidad marginal de U.

   (c) Hallar $f_{T|U=1}(t)$.

2. (25 pts.) Sean U e Y variables aleatorias independientes con $V(U) = 2$, $V(Y) = 3$.

   (a) Hallar $cov(2Y - U, U - Y)$.

   (b) Sea $X = 3U + 2Y$. Hallar $\rho(U, X)$.

   (c) Si además $E(U) = 1$, $E(Y) = 3$. Hallar $V(UY)$.

3. (25 pts.) Sean $X_1, \ldots, X_n$ variables aleatorias i.i.d con función de densidad

$$f_X(x;\theta) = \frac{3}{\theta}\left(\frac{1}{2} - \frac{2x^2}{\theta^2}\right)\mathbb{I}_{[-\frac{\theta}{2},\frac{\theta}{2}]}(x), \qquad (\theta > 0).$$

   (a) Hallar el estimador de momentos de θ.

   (b) Definir consistencia de un estimador y probar que el estimador hallado es consistente.

   (c) Si $n = 10$, $\overline{X} = 0.1$, $s^2 = 4.44$. Calcular $\hat{\theta}$.

4. (25 puntos)

(a) Sea $X_1, \ldots, X_n$ una m.a. de variables aleatorias $Bi(1,p)$, siendo n grande. Halle un test de nivel aproximado α para testear las hipótesis

$$H_0 : p = 0.30 \qquad\qquad H_1 : p < 0.30$$

y pruebe que el test propuesto tiene el nivel aproximado requerido.

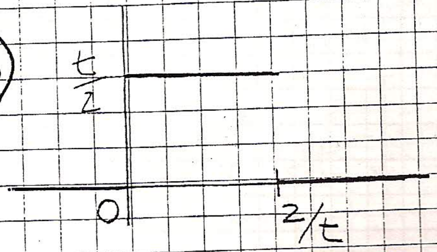
(b) Calcular la probabilidad de no rechazar H_0 cando en realidad $p = 0.26$.

(c) La proporción de fumadores en cierta ciudad era del 30%. Luego de una campaña publicitaria contra el cigarrillo se desea saber si ha tenido éxito. Se va arealizar un test de nivel aroximado 0.05. ¿Cuál deberá ser el tamaño de la muestra si se quiere que la probabilidad de error de tipo II cuando la verdadera proporción de fumadores bajó al 26 % sea ≤ 0.10?

EX. FINAL 19/03/04

1) $T \sim \Gamma(2,2) \qquad U/_{T=t} \sim U\left[0, \frac{2}{t}\right]$

b)



$$f_{U/_{T=t}}(u) = \begin{cases} \frac{t}{2} & \text{si } 0 \leq u \leq \frac{2}{t} \\ 0 & \text{fuera} \end{cases}$$

$$f_{U/_{T=t}}(u) = \frac{f_{UT}(u,t)}{f_T(t)}$$

$$\Rightarrow f_{UT}(u,t) = f_{U/_{T=t}}(u) \cdot f_T(t)$$

X tiene distrib. $\Gamma(\alpha, \lambda)$ $\quad \alpha > 0, \lambda > 0$

$$f_X(x) = \frac{e^{-\lambda x} \cdot x^{\alpha-1} \cdot \lambda^{\alpha}}{\Gamma(\alpha)} I_{(0,\infty)}(x)$$

• Si $\alpha \in \mathbb{N} \Rightarrow \Gamma(\alpha) = (\alpha-1)!$

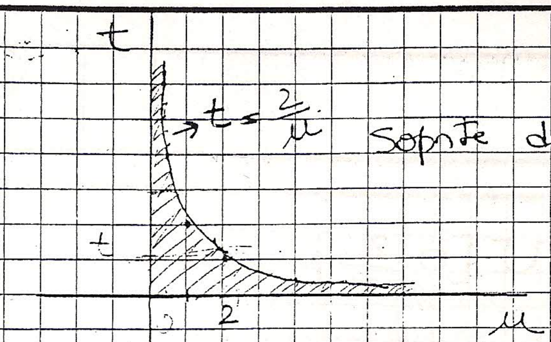
$T \sim \Gamma(2,2)$

$$f_T(t) = \frac{e^{-2t} \cdot t^{2-1} \cdot 2^2}{(2-1)!} I_{(0,\infty)}(t) = 4 t e^{-2t} \cdot I_{(0,\infty)}(t)$$

luego

$$f_{UT}(u,t) = \frac{t}{2} \cdot I_{(0,\frac{2}{t})}(u) \cdot 4 t \cdot e^{-2t} \cdot I_{(0,\infty)}(t) =$$

$$= 2 t^2 e^{-2t} \cdot I_{(0, 2/t)}(u) \cdot I_{(0,\infty)}(t)$$



Soporte de la densidad f_{UT} conjunta

$$f_U(u) = \int_{-\infty}^{+\infty} f_{UT}(u,t)\,dt =$$

Sea $u>0$ (si $u<0$ $f_{UT}(u,t)=0$)

$$f_U(u) = \int_0^{2/u} 2t^2 \cdot e^{-2t}\,dt =$$

$$= 2\int_0^{2/u} t^2 e^{-2t}\,dt$$

$$\int t^2 e^{-2t}\,dt = -\frac{1}{2}t^2 \cdot e^{-2t} + \underbrace{\int t \cdot e^{-2t}\,dt}_{*}$$

$u = t^2$ $\quad u' = 2t$

$v' = e^{-2t}$ $\quad v = -\frac{1}{2}e^{-2t}$

$$\int t\, e^{-2t}\,dt = -\frac{1}{2}t \cdot e^{-2t} + \frac{1}{2}\int e^{-2t}\,dt =$$

*

$u = t$ $\quad u' = 1$

$v' = e^{-2t}$ $\quad v = -\frac{1}{2}e^{-2t}$

$$= -\frac{1}{2}te^{-2t} + \frac{1}{2}\left(-\frac{1}{2}e^{-2t}\right) =$$

$$= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}$$

$$\int t^2 e^{-2t}\, dt = -\frac{1}{2} t^2 \cdot e^{-2t} - \frac{1}{2} t\, e^{-2t} - \frac{1}{4} e^{-2t}$$

$$\int_0^{2/u} t^2 e^{-2t}\, dt = -\frac{1}{2} \cdot \frac{4}{u^2}\, e^{-2\cdot\frac{2}{u}} - \frac{1}{2} \cdot \frac{2}{u} \cdot e^{-2\cdot\frac{2}{u}} -$$

$$- \frac{1}{4}\, e^{-2\cdot 2/u} - \left(-\frac{1}{4}\right) =$$

$$= -\frac{2}{u^2}\, e^{-4/u} - \frac{1}{u}\, e^{-4/u} - \frac{1}{4}\, e^{-4/u} + \frac{1}{4}$$

$$f_U(u) = \left(-\frac{4}{u^2}\, e^{-4/u} - \frac{2}{u}\, e^{-4/u} - \frac{1}{2}\, e^{-4/u} + \frac{1}{2}\right) I_{(0,+\infty)}(u)$$

a) $$f_{U/T=t}(u) = \frac{f_{UT}(u,t)}{f_T(t)}$$

$$f_{U/T=t}(u) = \begin{cases} t/2 & \text{si} \quad 0 \leq u \leq \frac{2}{t} \\ 0 & \text{fuera} \end{cases}$$

$$f_{UT}(u,t) = f_{U/T=t}(u)\; f_T(t)$$

$$f_U(u) = \int_{-\infty}^{+\infty} f_{UT}(u,t)\, dt =$$

$$= \int_{-\infty}^{+\infty} f_{U/T=t}(u)\; f_T(t)\, dt$$

$$E(U) = \int_{-\infty}^{+\infty} u \left(\int_{-\infty}^{+\infty} \left[\frac{t}{2} I_{(0,\,2/t)}(u) \cdot 4te^{-2t} \cdot I_{(0,+\infty)}(t) \right] dt \right) du =$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} u \cdot \frac{t}{2} \, I_{(0,\,2/t)}(u) \cdot 4t \cdot e^{-2t} \cdot I_{(0,+\infty)}(t) \, dt \right) du =$$

$$0 < t < \infty$$

$$0 < u < \frac{2}{t}$$

$$= \int_{0}^{\infty} \left(\int_{0}^{2/t} u \cdot 2\underline{t^2} e^{-2t} \, du \right) dt =$$

$$= \int_{0}^{\infty} t^2 e^{-2t} \left(\int_{0}^{2/t} 2u \, du \right) dt =$$

$$= \int_{0}^{\infty} t^2 e^{-2t} \left(u^2 \Big|_{0}^{2/t} \right) dt =$$

$$= \int_{0}^{\infty} \cancel{t^2} e^{-2t} \left(\frac{4}{\cancel{t^2}} \right) dt =$$

$$= 4 \int_{0}^{\infty} e^{-2t} \, dt = 4 \left(-\frac{1}{2} e^{-2t} \Big|_{0}^{\infty} \right) =$$

$$= 4 \cdot \left(0 + \frac{1}{2} \right) = \boxed{2}$$

c) $f_{T/U=1}(t) = \dfrac{f_{UT}(1,t)}{f_U(1)}$

$f_U(1) = -4\,e^{-4} - 2\,e^{-4} - \frac{1}{2}\,e^{-4} + \frac{1}{2} =$

$= -6{,}5\,e^{-4} + \frac{1}{2}$

$f_{UT}(1,t) = 2 \cdot t^2\,e^{-2t} \cdot I_{(0,2)}(t)$

si $1 < \frac{2}{t}$ y $t \in (0,\infty)$

$t < 2$

$f_{T/U=1}(t) = \dfrac{2\,t^2\,e^{-2t}}{-6{,}5\,e^{-4} + 1/2} \cdot I_{(0,2)}(t)$

2) $V(U) = 2$ $\quad V(Y) = 3$

a) $Cov(2Y - U, U - Y) =$

$= E\big((2Y-U)(U-Y)\big) - \left[E(2Y-U) \cdot E(U-Y)\right] =$

$= E\left(2UY - 2Y^2 - U^2 + UY\right) - \left(\left[E(2Y) - E(U)\right]\left[E(U) - E(Y)\right]\right)$

$= 2\,E(U)\,E(Y) - 2\,E(Y^2) - E(U^2) + E(U)\,E(Y) -$

$- \left[2\,E(U)\,E(Y) - 2\,E(Y)^2 - E(U)^2 + E(U)\,E(Y)\right] =$

$= 2\,\cancel{E(U)\,E(Y)} - 2\,E(Y^2) - E(U^2) + \cancel{E(U)\cdot E(Y)}$

$- 2\,\cancel{E(U)\,E(Y)} + 2\,E(Y)^2 + E(U)^2 - \cancel{E(U)\cdot E(Y)} =$

$= -2\left[E(Y^2) - E(Y)^2\right] + (-1)\left[E(U^2) - E(U)^2\right] =$

$= -2\cdot V(Y) - V(U) = -2\cdot 3 - 2 = \boxed{-8}$

b) $X = 3U + 2Y \qquad \rho(U,X) = \dfrac{Cov(U,X)}{\sigma_U \cdot \sigma_X}$

$Cov(U,X) = E(UX) - E(U)\,E(X) =$

$= E(U(3U+2Y)) - E(U)\cdot E(3U+2Y) =$

$= E(3U^2 + 2UY) - E(U)\left[E(3U) + E(2Y)\right] =$

$= 3\cdot E(U^2) + 2\,E(U)\cdot E(Y) - E(U)\left[3\,E(U) + 2\,E(Y)\right] =$

$= 3\,E(U^2) + 2\,\cancel{E(U)\,E(Y)} - 3\,E(U)^2 - 2\,\cancel{E(U)\cdot E(Y)} =$

$= 3\left[E(U^2) - E(U)^2\right] = 3\cdot V(U) = \boxed{6}$

$\sigma_U = \sqrt{V(U)} = \sqrt{2}$

$V(X) = V(3U+2Y) = E\left((3U+2Y)^2\right) - \left[E(3U+2Y)\right]^2 =$

$= E\left(9U^2 + 12UY + 4Y^2\right) - \left[3E(U) + 2E(Y)\right]^2 =$

$= 9\,E(U^2) + 12\,\cancel{E(U)\,E(Y)} + 4\,E(Y^2) - 9\,E(U)^2$

$- 12\,\cancel{E(U)\,E(Y)} - 4\,E(Y)^2 =$

$= 9\cdot V(U) + 4\,V(Y) = 18 + 4\cdot 3 = 30$

$\sigma_X = \sqrt{30} \qquad \rho(U,X) = \dfrac{6}{\sqrt{60}}$

c) $E(U) = 1 \quad E(Y) = 3 \quad V(UY) = ?$

$$V(UY) = E\left((UY)^2\right) - [\underbrace{E(UY)}_{3}]^2 =$$

$$= E\left((UY)^2\right) - 9$$

Si U y V son indep $\Rightarrow$ U^2 y V^2 tamb lo son

$$= E(U^2 \cdot Y^2) - 9 =$$

$$= E(U^2) \cdot E(Y^2) - 9 =$$

$$= 3 \cdot 12 - 9 = 27$$

$$V(U) = E(U^2) - E(U)^2 \Rightarrow E(U^2) = V(U) + E(U)^2$$

$$= 2 + 1 = 3$$

$$E(Y^2) = V(Y) + E(Y)^2 =$$

$$= 3 + 9 = 12$$

③ $f_X(x,\theta) = \frac{3}{\theta}\left(\frac{1}{2} - \frac{2x^2}{\theta^2}\right) I_{\left[-\frac{\theta}{2};\frac{\theta}{2}\right]}(x)$ $(\theta>0)$

a) Estimador de momentos de θ

$$E(X) = \int_{-\infty}^{+\infty} x\,\frac{3}{\theta}\left(\frac{1}{2} - \frac{2x^2}{\theta^2}\right) I_{\left[-\frac{\theta}{2};\frac{\theta}{2}\right]}(x)\, dx =$$

$$= \frac{3}{\theta}\int_{-\frac{\theta}{2}}^{\theta/2} \frac{1}{2}x - 2\frac{x^3}{\theta^2}\, dx =$$

$$= \frac{3}{\theta}\left(\int_{-\frac{\theta}{2}}^{\theta/2} \frac{1}{2}x\, dx - \int_{-\theta/2}^{\theta/2} 2\frac{x^3}{\theta^2}\, dx\right) =$$

$$= \frac{3}{\theta}\left(\frac{1}{2}\underbrace{\left(\frac{x^2}{2}\Big|_{-\theta/2}^{\theta/2}\right)}_{=0} - \frac{2}{\theta^2}\underbrace{\left(\frac{x^4}{4}\Big|_{-\frac{\theta}{2}}^{\theta/2}\right)}_{=0}\right) =$$

$E(X) = 0 \rightarrow$ no obtengo información

Paso al 2do momento.

$$V(X) = E(X^2) - \underbrace{E(X)^2}_{=0}$$

$$E(X^2) = V(X) = \int_{-\frac{\theta}{2}}^{\theta/2} x^2\,\frac{3}{\theta}\left(\frac{1}{2} - 2\frac{x^2}{\theta^2}\right) dx =$$

$$= \frac{3}{\theta}\int_{-\frac{\theta}{2}}^{\theta/2} \frac{1}{2}x^2 - 2\frac{x^4}{\theta^2}\, dx =$$

$$= \frac{3}{\theta}\left(\frac{1}{2}\int_{-\frac{\theta}{2}}^{\theta/2} x^2\, dx - \frac{2}{\theta^2}\int_{-\frac{\theta}{2}}^{\theta/2} x^4\, dx\right) =$$

$$= \frac{3}{\theta}\left(\frac{1}{2}\cdot\frac{X^3}{3}\Big|_{-\theta/2}^{\theta/2} - \frac{2}{\theta^2}\,\frac{X^5}{5}\Big|_{-\theta/2}^{\theta/2}\right) =$$

$$= \frac{3}{\theta}\left(\frac{1}{2}\left(\frac{\theta^3}{24}+\frac{\theta^3}{24}\right) - \frac{2}{\theta^2}\left(\frac{\theta^5}{160}+\frac{\theta^5}{160}\right)\right) =$$

$$= \frac{3}{\theta}\cdot\left(\frac{1}{2}\,\frac{\theta^3}{12} - \frac{2}{\theta^2}\,\frac{\theta^5}{80}\right) = \frac{3}{24}\theta^2 - \frac{6}{80}\theta^2 = \frac{\theta^2}{20}$$

$$E(X^2) = \frac{\theta^2}{20} \simeq \frac{\sum_{i=1}^{n} X_i^2}{n}$$

$$\hat{\theta} = \sqrt{20\cdot\frac{\sum_{i=1}^{n} X_i^2}{n}}$$ es el estimador de momentos de $\hat{\theta}$

b) Consistencia

$\hat{\theta}_n$ es un estimador consistente de θ si

$$\underbrace{\hat{\theta}_n}_{\downarrow} \xrightarrow{P} \theta$$

estim. puntual
basado en una
m.a. de una distrib.

Por la LGN $$\frac{\sum_{i=1}^{n} X_i^2}{n} \xrightarrow{P} \underbrace{E(X^2)}_{\frac{\theta^2}{20}}$$

luego $$20\cdot\frac{\sum_{i=1}^{n} X_i^2}{n} \xrightarrow{P} \theta^2$$

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$$y \quad \underbrace{\sqrt{\frac{20 \cdot \sum_{i=1}^{n} x_i^2}{n}}}_{\hat{\theta}} \xrightarrow{P} \theta$$

c) $n = 10 \quad \overline{X} = 0{,}1 \quad S^2 = 4{,}44 \quad \hat{\theta} = ?$

$$S^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n-1} = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(x_i^2 - 2x_i\overline{x} + \overline{x}^2\right)\right) =$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - 2\overline{x} \cdot \sum_{i=1}^{n} x_i + n \cdot \overline{x}^2\right) =$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - 2 \cdot \overline{x} \cdot \overline{x}\, n + n\,\overline{x}^2\right) =$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n \cdot \overline{x}^2\right) =$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n} x_i^2 - \frac{n}{n-1}\,\overline{x}^2$$

$$4{,}44 = \frac{1}{9}\sum_{i=1}^{n} x_i^2 - \frac{10}{9}\cdot(0{,}1)^2$$

$$4{,}44 = \frac{1}{9}\sum_{i=1}^{n} x_i^2 - \frac{1}{90}$$

$$\left(4{,}44 + \frac{1}{90}\right) \cdot 9 = \sum_{i=1}^{n} x_i^2$$

$$39{,}96 + 0{,}1 = \sum_{i=1}^{n} x_i^2$$

$$40{,}06 = \sum_{i=1}^{n} x_i^2$$

$$\hat{\theta} = \sqrt{\frac{20 \cdot 40{,}06}{10}}$$

$$\hat{\theta} = \sqrt{80{,}12} \simeq 8{,}951$$

(4) $X_1, X_2 \dots X_n$ m.a. $Bi(1,p)$

luego $\sum_{i=1}^{n} X_i \sim Bi(n,p)$

Por el teo central del límite

$$\frac{\bar{X} - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} Z \sim N(0,1)$$

El test de nivel aprox α para las hipótesis

$H_0: p = 0.30$ $\qquad H_1: p < 0.30$

se basa en el estadístico

$$\frac{\bar{X} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = Z_{obs}$$

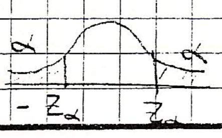
Rechazo H_0) si $Z_{obs} \leq -Z_\alpha$

Voy a probar que el test tiene el nivel aprox. requerido.

$P(\text{rechazar } H_0 / \text{es verdadera}) = \alpha$

$P(\text{rechazar } H_0 / \text{es verdadera}) = P(\text{rechazar } H_0 \text{ cuando } p = 0{,}3$

$$= P\Bigg(\underbrace{\frac{\bar{X} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}}_{\substack{\text{bajo } H_0 \text{ tiene} \\ \text{dist } N(0,1)}} \leq -Z_\alpha \Bigg) = \alpha$$



b) $P(\text{error tipo II}) = P(\text{aceptar } H_0 / \text{es falsa}) =$

$$= P\left(\frac{\bar{X} - p_0}{\sqrt{\frac{p_0(1-p_0)}{m}}} > -Z_\alpha \;\Big/\; p = 0{,}26 \right) =$$

$$= P\left(\bar{X} > -Z_\alpha \sqrt{\frac{p_0(1-p_0)}{m}} + p_0 \;\Big/\; p = 0{,}26 \right) =$$

$$= P\left(\underbrace{\frac{\bar{X} - \overset{p}{0{,}26}}{\sqrt{\frac{0{,}26 \cdot 0{,}74}{m}}}}_{\sim N(0,1)} > \frac{-Z_\alpha \sqrt{\frac{p_0(1-p_0)}{m}} + p_0 - \overset{p}{0{,}26}}{\sqrt{\frac{0{,}26 \cdot 0{,}74}{m}}} \right)$$

c) $p = 0{,}26$

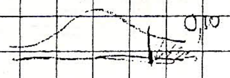
$H_0)\; p = \underbrace{0{,}30}_{p_0}$ $\qquad$ $H_1)\; p < \underbrace{0{,}30}_{p_0}$

$\alpha = 0{,}05$ $\qquad$ $Z_{0{,}05} = 1{,}64$ $\qquad$ $p_0 = 0{,}30$

$$P(\text{error tipo II}) = P\left(Z > \frac{\frac{-1{,}64\sqrt{0{,}30 \cdot 0{,}70}}{\sqrt{m}} + 0{,}30 - 0{,}26}{\frac{\sqrt{0{,}26 \cdot 0{,}74}}{\sqrt{m}}} \right) \leq 0{,}10$$

$$P\left(Z > \frac{\frac{-0{,}752}{\sqrt{m}} + 0{,}04}{\frac{0{,}439}{\sqrt{m}}} \right) \leq 0{,}10$$

$$P\left(Z > \frac{-0{,}752}{0{,}439} + \frac{\sqrt{m} \cdot 0{,}04}{0{,}439} \right) \leq 0{,}10$$



$$\frac{-0{,}752}{0{,}439} + \frac{\sqrt{m}\cdot 0{,}04}{0{,}439} \geqslant 1{,}29$$

$$\sqrt{m}\,\frac{0{,}04}{0{,}439} \geqslant 1{,}29 + \frac{0{,}752}{0{,}439}$$

$$\sqrt{m} \geqslant 32{,}96$$

$$\boxed{m \geqslant 1087}$$

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